ON THE COMPLETENESS OF GENERALIZED EIGENFUNCTIONS OF ELLIPTIC CONE OPERATORS

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Abstract. We show the completeness of the system of generalized eigenfunctions of closed extensions of elliptic cone operators under suitable conditions on the symbols.

1. Introduction

The purpose of this note is to extend the theorem about completeness of the system of generalized eigenfunctions of elliptic operators on manifolds with conical singularities of Egorov, Kondratiev, and Schulze [7, 8] to the general case. While it is implicit in their presentation, it is, however, important to note that their result is applicable only for the minimal extension of the operator. This leaves out many important cases, including (nonselfadjoint) realizations of Laplacians. We will present two simple examples in Section 5 which illustrate the relevancy of this observation.

Like Egorov, Kondratiev, and Schulze, we will follow Agmon's approach [1] towards proving this result. This approach is based on a purely functional analytic theorem of Dunford and Schwartz [6, Chapter XI.9 and XI.6], see Section 2, which reduces the task of proving completeness of generalized eigenfunctions to showing that the embedding of the domain of the operator into the Hilbert space is of Schatten class, and to showing that the operator admits sufficiently many rays of minimal growth. Agranovich uses the same approach in [2, Section 6.4] and [3, Section 9.3] to address the completeness problem for elliptic operators on smooth manifolds.

Rays of minimal growth for elliptic cone operators equipped with general domains have been the subject of our earlier work [10, 11, 12] in collaboration with J. Gil and G. Mendoza (for the boundaryless case), and [16] (for the case of realizations subject to boundary conditions). With these results at hand, it remains to prove that the embedding of the domain of the operator into the Hilbert space is of Schatten class. To do this, we will employ recent results of Buzano and Toft [4, 21] as well as the explicit descriptions of domains of elliptic cone operators from [17, 15] (in the boundaryless case) and [5, 16] (for realizations subject to boundary conditions).

The focus of Agmon's original paper [1] are elliptic boundary value problems on smooth manifolds. He uses Fourier series to show that the embeddings of the Sobolev spaces are of Schatten class. This argument was adapted by Egorov, Kondratiev, and Schulze in [7, 8]. In [2, 3], Agranovich uses a different elegant argument to show that the embedding of the domain of an operator A is of Schatten class; this argument is based on the Weyl asymptotics of the eigenvalues of the operator $(A-\lambda_0)(A-\lambda_0)^*$ for a suitable λ_0 in the resolvent set of A. That semibounded

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elliptic cone operators in the boundaryless case exhibit Weyl asymptotics has been shown by Lesch [17]. In the boundaryless case, we could therefore follow Agranovich's argument to obtain what is needed to prove completeness of the generalized eigenfunctions. However, our approach gives a more general embedding result which readily applies to all kinds of realizations of elliptic cone operators on conic manifolds with or without boundary.

The structure of the paper is as follows:

In Section 2 we review the functional analytic background and the result of Dunford and Schwartz [6, Chapter XI.9 and XI.6].

Section 3 is devoted to weighted cone Sobolev spaces [19, 20] and the embedding result that we need.

Section 4 summarizes basics about elliptic cone operators, and we review the results about rays of minimal growth from [10, 11, 12, 16].

We conclude this work in Section 5 with the main theorems about the completeness of generalized eigenfunctions for general realizations of elliptic cone operators, and the discussion of two simple examples to illustrate the results.

The case of cone operators represents the simplest situation of elliptic operators on incomplete Riemannian manifolds with corners. From this perspective, this work is the first step towards addressing similar questions for this more general case. The observations made in the present work will impact such future investigations. As the examples in Section 5 show, it cannot be expected that the scales of weighted Sobolev spaces that are considered in the existing literature on elliptic operators on incomplete manifolds with corners will be immediately related to the functional analytic domains of an elliptic operator. Much work still needs to be done to describe these domains. On the other hand, the present paper underscores that the proof of the completeness of generalized eigenfunctions is based only on very few principles.

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2. Functional analytic background

Let H be a complex Hilbert space, and let

$$A : \mathcal{D} \subset H \to H$$

be a closed, densely defined operator acting in H. The domain $\mathcal{D}$ is equipped with the graph norm. Having realizations of elliptic operators in mind, we usually write $A_{\mathcal{D}}$ to emphasize that A acts in H with domain $\mathcal{D}$.

Recall that a vector $0 \neq u \in H$ is called a *generalized eigenvector* of $A_{\mathcal{D}}$ associated with the eigenvalue $\lambda_0 \in \mathbb{C}$ if $(A_{\mathcal{D}} - \lambda_0)^k u = 0$ for some $k \geq 1$. This entails of course that u is in the domain of the k-th power of $A_{\mathcal{D}}$. Let $\mathfrak{Eig}(A_{\mathcal{D}})$ denote the linear span of all generalized eigenvectors of $A_{\mathcal{D}}$.

The statement that the system of generalized eigenvectors is complete in H means that $\mathfrak{Eig}(A_{\mathcal{D}})$ is dense in H.

Theorem 2.1 ([6, Corollary XI.9.31])**.** *Suppose the embedding $\mathcal{D} \hookrightarrow H$ belongs to the Schatten class $\mathfrak{S}_p$ for some $0 < p < \infty$. Moreover, let there be rays*

$$\Gamma_j = \{re^{i\theta_j};\ r \geq 0\}, \quad j = 1, \dots, J,$$

in the complex plane that are rays of minimal growth for the operator $A_{\mathcal{D}}$, and such that all angles enclosed by any two adjacent rays are $\leq \pi/p$.

Then the system of generalized eigenvectors of $A_{\mathcal{D}}$ is complete in H.

Recall that a ray $\Gamma = \{re^{i\theta};\ r \geq 0\} \subset \mathbb{C}$ is called a *ray of minimal growth* or a *ray of maximal decay* for A if

$$A - \lambda : \mathcal{D} \to H$$

is invertible for $\lambda \in \Gamma$ with $|\lambda| > 0$ sufficiently large, and if the resolvent satisfies the estimate

$$\|(A_{\mathcal{D}} - \lambda)^{-1}\|_{\mathscr{L}(H)} = O\big(|\lambda|^{-1}\big)$$

as $|\lambda| \to \infty$ in Γ.

Moreover, for Hilbert spaces E and F, $\mathfrak{S}_p$ is the space of all $T \in \mathscr{L}(E,F)$ such that $\sum_{j=0}^{\infty} \alpha_j(T)^p < \infty$, where

$$\alpha_j(T) = \inf\{\|T - G\|_{\mathscr{L}(E,F)};\ G \in \mathscr{L}(E,F),\ \dim R(G) \leq j\}$$

is the j-th approximation number of T.

Observe that if Γ is a ray of minimal growth, then there is a sector Λ with $\mathring{\Lambda} \neq \emptyset$ and central axis Γ such that all rays in Λ are rays of minimal growth for $A_{\mathcal{D}}$. This implies that in Theorem 2.1 above we can weaken the assumption to only require that the embedding $\mathcal{D} \hookrightarrow H$ belongs to $\mathfrak{S}_p^+$, where

$$\mathfrak{S}_p^+ = \bigcap_{q>p} \mathfrak{S}_q. \tag{2.2}$$

Note that $\mathfrak{S}_i \subset \mathfrak{S}_j$ for $0 < i < j < \infty$. This observation is rather useful when dealing with elliptic operators since the embeddings of domains typically belong to $\mathfrak{S}_p^+$, where $p > 0$ depends on the order of the operator and the dimension of the underlying space (see below).

If the operator $A_{\mathcal{D}}$ has nonempty resolvent set $\varrho(A_{\mathcal{D}})$ as is assumed in Theorem 2.1, the condition that the embedding $\mathcal{D} \hookrightarrow H$ belongs to the Schatten class $\mathfrak{S}_p$ for some $0 < p < \infty$ is equivalent to requiring that the resolvent $T = (A_{\mathcal{D}} - \lambda_0)^{-1} : H \to H$ belongs to $\mathfrak{S}_p$ for some $\lambda_0 \in \varrho(A_{\mathcal{D}})$. Recall that this means that the nonzero eigenvalues (counting multiplicities) $\lambda_0(\sqrt{T^*T}) \geq \lambda_1(\sqrt{T^*T}) \geq \ldots > 0$ of $\sqrt{T^*T}$ are p-summable, i.e., $\sum_{j=0}^{\infty} \lambda_j(\sqrt{T^*T})^p < \infty$.

In view of the identity $\sqrt{T^*T} = \big[(A_{\mathcal{D}} - \lambda_0)(A_{\mathcal{D}} - \lambda_0)^*\big]^{-1/2}$ and the spectral theorem for selfadjoint operators, we conclude that if $A_{\mathcal{D}}$ has compact resolvent and the eigenvalues $0 < \mu_0 \leq \mu_1 \leq \ldots$ of $(A-\lambda_0)(A-\lambda_0)^*$ (counting multiplicities) obey Weyl's law $\mu_j \sim \mathrm{Const} \cdot j^{\frac{2m}{n}}$ as $j \to \infty$, then T belongs to $\mathfrak{S}_{n/m}^+$. Here $m, n > 0$, and in applications to elliptic operators m is the order of A and n the dimension of the underlying space. This is Agranovich's argument from [2, 3] to prove that the embeddings of domains of elliptic operators on smooth compact manifolds are of Schatten class. As already mentioned in the introduction, we will follow in this paper a different approach for realizations of elliptic cone operators.

3. Embeddings of weighted cone Sobolev spaces

We begin with a brief review of the definition of the scale of weighted b-Sobolev spaces. More details can be found in [19, 20].

Let $\overline{M}$ be a smooth, compact n-manifold with boundary $\partial\overline{M}$, and let $x \in C^\infty(\overline{M})$ be a defining function for $\partial\overline{M}$. Recall that this means that $x \geq 0$ on $\overline{M}$, $\partial\overline{M} = \{x = 0\}$, and $dx \neq 0$ on $\partial\overline{M}$. By $L^2_b(\overline{M})$ we denote the L^2-space with respect to any b-density $\mathfrak{m}$ on $\overline{M}$. Recall that $\mathfrak{m}$ is a b-density if $x\mathfrak{m}$ is a smooth, everywhere positive density on $\overline{M}$. The b-Sobolev space of smoothness $s \in \mathbb{N}_0$ is defined as

$$H^s_b(\overline{M}) = \{u \in \mathcal{D}'(\overline{M});\ Pu \in L^2_b(\overline{M}) \text{ for all } P \in \mathrm{Diff}^m_b(\overline{M}),\ m \leq s\}.$$

Recall that $\mathrm{Diff}^m_b(\overline{M})$ is the space of b-differential operators of order m, i.e., the operators of order m in the enveloping algebra of differential operators generated by $C^\infty(\overline{M})$ and the Lie algebra $\mathcal{V}_b$ of smooth vector fields on $\overline{M}$ that are tangential to the boundary. For general $s \in \mathbb{R}$ the space $H^s_b(\overline{M})$ is defined by interpolation and duality. More generally, if E is a (Hermitian) vector bundle on $\overline{M}$, let $x^\gamma H^s_b(\overline{M}; E)$ be the weighted b-Sobolev space of sections of E of regularity $s \in \mathbb{R}$.

Our first goal in this section is the following theorem.

Theorem 3.1. *The embedding*

$$x^\gamma H^s_b(\overline{M}; E) \hookrightarrow x^{\gamma'} H^{s'}_b(\overline{M}; E)$$

belongs to the Schatten class $\mathfrak{S}_p$, $0 < p < \infty$, *for any* $\gamma > \gamma'$ *and* $s > s' + n/p$.

The proof of Theorem 3.1 makes use of a corresponding result about embeddings of weighted Sobolev spaces on $\mathbb{R}^n$. More precisely, for $s, \delta \in \mathbb{R}$ let $H^{s,\delta}(\mathbb{R}^n) = \langle x\rangle^{-\delta} H^s(\mathbb{R}^n)$ (unlike in other contexts in this paper, x represents the variable in $\mathbb{R}^n$ here). We have the following lemma.

Lemma 3.2. *The embedding*

$$\iota : H^{s,\delta}(\mathbb{R}^n) \hookrightarrow H^{s',\delta'}(\mathbb{R}^n)$$

belongs to the Schatten class $\mathfrak{S}_p$, $0 < p < \infty$, *for any* $\delta > \delta' + n/p$ *and* $s > s' + n/p$.

Proof. For the proof we may without loss of generality assume that $p > 1$: Otherwise, let $N \in \mathbb{N}$ with $\frac{1}{N} < p$, and consider the composition of embeddings

$$H^{s_0,\delta_0}(\mathbb{R}^n) \hookrightarrow H^{s_1,\delta_1}(\mathbb{R}^n) \hookrightarrow \ldots \hookrightarrow H^{s_N,\delta_N}(\mathbb{R}^n),$$

where $s_j = s - j \cdot \frac{s-s'}{N}$, $\delta_j = \delta - j \cdot \frac{\delta-\delta'}{N}$, $j = 0, \ldots, N$. In view of $s_{j-1} - s_j = \frac{s-s'}{N} > \frac{n}{Np}$ and $\delta_{j-1} - \delta_j = \frac{\delta-\delta'}{N} > \frac{n}{Np}$ and $Np > 1$ we may conclude that the embedding $H^{s_{j-1},\delta_{j-1}}(\mathbb{R}^n) \hookrightarrow H^{s_j,\delta_j}(\mathbb{R}^n)$ belongs to $\mathfrak{S}_{Np}$ (if we take the result of the lemma for granted for class indices greater than one). The composition of N mappings of class $\mathfrak{S}_{Np}$ belongs to $\mathfrak{S}_p$ by the general properties of these classes.

Hence assume in the sequel that $p > 1$. For $\mu, \varrho \in \mathbb{R}$ let $\Lambda^{\mu,\varrho} = \langle x\rangle^\varrho \langle D_x\rangle^\mu$ and $\tilde{\Lambda}^{\mu,\varrho} = \langle D_x\rangle^\mu \langle x\rangle^\varrho$. Then

$$\Lambda^{\mu,\varrho}, \tilde{\Lambda}^{\mu,\varrho} : H^{s,\delta}(\mathbb{R}^n) \to H^{s-\mu,\delta-\varrho}(\mathbb{R}^n)$$

are isomorphisms for all $s, \delta \in \mathbb{R}$, and obviously $\left(\Lambda^{\mu,\varrho}\right)^{-1} = \tilde{\Lambda}^{-\mu,-\varrho}$. In view of the commutative diagram

$$\begin{array}{ccc} H^{s,\delta}(\mathbb{R}^n) & \xrightarrow{\iota} & H^{s',\delta'}(\mathbb{R}^n) \\ {\scriptstyle \Lambda^{s',\delta'}}\big\downarrow & & \big\uparrow{\scriptstyle \tilde{\Lambda}^{-s',-\delta'}} \\ H^{s-s',\delta-\delta'}(\mathbb{R}^n) & \xrightarrow{\iota} & L^2(\mathbb{R}^n) \end{array}$$

and the operator ideal property of the Schatten classes $\mathfrak{S}_p$ we may assume without loss of generality that $s' = \delta' = 0$. Using again the operator ideal property and the commutative diagram

$$\begin{array}{ccc} H^{s,\delta}(\mathbb{R}^n) & \xrightarrow{\iota} & L^2(\mathbb{R}^n) \\ {\scriptstyle \tilde{\Lambda}^{s,\delta}}\big\downarrow & & \big\| \\ L^2(\mathbb{R}^n) & \xrightarrow{\Lambda^{-s,-\delta}} & L^2(\mathbb{R}^n) \end{array}$$

we see that it suffices to show that the operator $\Lambda^{-s,-\delta} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ belongs to $\mathfrak{S}_p$ for $s > n/p$ and $\delta > n/p$. This, however, is a direct consequence of [4, Proposition 4.2], it also follows from [21] (these papers consider Schatten classes with indices $p \geq 1$, this is why we made that reduction at the beginning of this proof). The point here is that the symbol $\langle x\rangle^{-\delta}\langle\xi\rangle^{-s}$ of the operator $\Lambda^{-s,-\delta}$ belongs to $L^p(\mathbb{R}^{2n})$ precisely if $s > n/p$ and $\delta > n/p$. The papers [4, 21] are concerned with characterizing the Schatten class property with index p for certain classes of pseudodifferential operators acting on $L^2(\mathbb{R}^n)$ in terms of L^p-bounds on their symbols (or on the weight functions of the symbol classes). In our situation at hand these results are applicable and lead to the desired conclusion. $\square$

Proof of Theorem 3.1. Since the proof is based on a localization argument, we may assume without loss of generality that $E \to \overline{M}$ is the trivial line bundle. Moreover, in view of the commutative diagram

$$\begin{array}{ccc} x^{\gamma} H_b^s(\overline{M}) & \xrightarrow{\iota} & x^{\gamma'} H_b^{s'}(\overline{M}) \\ {\scriptstyle x^{-\gamma'}}\big\downarrow & & \big\uparrow{\scriptstyle x^{\gamma'}} \\ x^{\gamma-\gamma'} H_b^s(\overline{M}) & \xrightarrow{\iota} & H_b^{s'}(\overline{M}) \end{array}$$

we may assume that $\gamma' = 0$.

Choose a collar neighborhood $\chi_0 : U_0 \cong [0,\varepsilon) \times \partial\overline{M}$ of the boundary. Without loss of generality, we may assume that the defining function x coincides in U_0 with the projection map to the coordinate in $[0,\varepsilon)$. Away from the boundary choose a finite collection $U_1, \ldots, U_N$ of open subsets of $\overline{M}$ that are via charts $\chi_j : U_j \to \Omega_j$, $j = 1, \ldots, N$, diffeomorphic to open bounded subsets $\Omega_j \subset \mathbb{R}^n$ such that $\overline{M} = \bigcup_{j=0}^N U_j$. Let φ_j, $j = 0, \ldots, N$, be a smooth partition of unity subordinate to this covering, and choose $\psi_j \in C^\infty(\overline{M})$ with compact support contained in U_j such that $\psi_j \equiv 1$ in a neighborhood of the support of φ_j.

With this data we further proceed to define maps $T_j : x^{\gamma} H_b^s(\overline{M}) \to H_b^{s'}(\overline{M})$, $j = 0, \ldots, N$, that belong to the Schatten class $\mathfrak{S}_p$ (provided that $s > s' + n/p$ as is assumed here), and such that $\iota = \sum\limits_{j=0}^{N} T_j$.

More precisely, for $j = 1, \ldots, N$ define T_j to be the composition of the maps

$$T_j = \big(\chi_j^* \circ \Psi_j\big) \circ \iota \circ \big(\chi_{j,*} \circ \varphi_j\big).$$

Here $\chi_{j,*} \circ \varphi_j : x^{\gamma} H_b^s(\overline{M}) \to H^{s,\delta}(\mathbb{R}^n)$ is the multiplication operator by φ_j followed by push-forward with respect to χ_j, where $\delta > n/p$ can be chosen arbitrarily. $\iota : H^{s,\delta}(\mathbb{R}^n) \to H^{s',0}(\mathbb{R}^n)$ is the embedding that belongs to $\mathfrak{S}_p$ by Lemma 3.2, and $\chi_j^* \circ \Psi_j : H^{s',0}(\mathbb{R}^n) \to H_b^{s'}(\overline{M})$ is the multiplication operator by $\Psi_j = \chi_{j,*}\psi$ followed by pull-back with respect to χ_j. All maps involved are continuous, and by

the operator ideal property of $\mathfrak{S}_p$ we obtain that T_j belongs to $\mathfrak{S}_p$ for $j = 1, \ldots, N$. Observe that $T_j u = \varphi_j u$ for $j = 1, \ldots, N$.

Analogously to the other T_j, the operator T_0 is just $T_0 u = \varphi_0 u$. In order to see that it belongs to $\mathfrak{S}_p$ we proceed as follows: Choose coordinate neighborhoods $U_{0j} \subset \partial\overline{M}$, $j = 1, \ldots, M$, and charts $\chi_{0j} : U_{0j} \to \Omega_{0j}$, where $\Omega_{0j} \subset \mathbb{R}^{n-1}$ is open and bounded, such that $\partial\overline{M} = \bigcup_{j=1}^{M} U_{0j}$. Choose a smooth subordinate partition of unity φ_{0j}, $j = 1, \ldots, M$, and functions $\psi_{0j} \in C^\infty(\partial\overline{M})$ with compact support in U_{0j} such that $\psi_{0j} \equiv 1$ in a neighborhood of the support of φ_{0j}. Let t be the diffeomorphism $\overline{\mathbb{R}}_+ \to \mathbb{R}$ defined by $t(x) = -\log(x)$. We write $T_0 = \sum_{j=1}^{M} T_{0j}$, where each operator T_{0j} is defined by

$$T_{0j} = \big[\big(\chi_0^* \circ \Psi_0\big) \circ \big((t,\chi_{0j})^* \circ \Psi_{0j}\big)\big] \circ \iota \circ \big[\big((t,\chi_{0j})_* \circ \varphi_{0j}\big) \circ \big(\chi_{0,*} \circ \varphi_0\big)\big]. \tag{3.3}$$

Here

$$\big[\big((t,\chi_{0j})_* \circ \varphi_{0j}\big) \circ \big(\chi_{0,*} \circ \varphi_0\big)\big] : x^\gamma H_b^s(\overline{M}) \to H^{s,\delta}(\mathbb{R}^n)$$

is continuous, where $\delta > n/p$ can be chosen arbitrarily. Indeed, multiplication by φ_0 and push-forward by χ_0 localizes distributions near the boundary and introduces the splitting of variables $(x, y) \in [0, \varepsilon) \times \partial\overline{M}$, multiplication by φ_{0j} localizes the y-dependence further to the coordinate neighborhood U_{0j}, push-forward by χ_{0j} in the y-variable and by t in the x-variable produces distributions on $\mathbb{R} \times \mathbb{R}^{n-1}$ that are supported in the strip $\mathbb{R} \times \Omega_{0j}$ and that vanish in a neighborhood of $t = -\infty$. The weight x^γ translates into an exponential weight $e^{-t\gamma}$ near $t = \infty$. In view of the support properties just discussed, we see that we certainly obtain a Sobolev distribution on $\mathbb{R}^n$ that exhibits any polynomial decay (in the Sobolev norm), or, in other words, we arrive in $H^{s,\delta}(\mathbb{R}^n)$ for any choice of $\delta > n/p$ as was claimed.

The other parts in (3.3) are the embedding $\iota : H^{s,\delta}(\mathbb{R}^n) \to H^{s',0}(\mathbb{R}^n)$ that belongs to $\mathfrak{S}_p$ by Lemma 3.2, and the operator

$$\big[\big(\chi_0^* \circ \Psi_0\big) \circ \big((t,\chi_{0j})^* \circ \Psi_{0j}\big)\big] : H^{s',0}(\mathbb{R}^n) \to H_b^{s'}(\overline{M})$$

consisting of multiplication by $\Psi_{0j} = \chi_{0j,*}\psi_{0j}$, pull-back via t and χ_{0j} and multiplication by $\Psi_0 = \chi_{0,*}\psi_0$ to yield disbritutions on $[0, \varepsilon) \times \partial\overline{M}$, and finally pull-back by χ_0 to yield distributions in $H_b^{s'}(\overline{M})$. Consequently, each operator T_{0j} belongs to $\mathfrak{S}_p$, and so $T_0 = \sum_{j=1}^{M} T_{0j}$ belongs to $\mathfrak{S}_p$.

In conclusion, $\iota = \sum_{j=0}^{N} T_j : x^\gamma H_b^s(\mathbb{R}^n) \to H_b^{s'}(\mathbb{R}^n)$ belongs to $\mathfrak{S}_p$ (provided that $s > s' + n/p$), and the proof of the theorem is complete. □

For the analysis of boundary value problems we will also need the corresponding version of Theorem 3.1 for the appropriate weighted b-Sobolev spaces on certain manifolds with corners.

More precisely, let $\overline{M}$ be a compact n-manifold with corners of codimension two (we work with the terminology from [19] here). Let $\partial\overline{M} = \partial_{\mathrm{reg}}\overline{M} \cup \partial_{\mathrm{sing}}\overline{M}$, where both $\partial_{\mathrm{reg}}\overline{M}$ and $\partial_{\mathrm{sing}}\overline{M}$ consist of unions of (different) boundary hypersurfaces of $\overline{M}$. We will refer to those hypersurfaces as regular or singular, respectively. We require that for any two hypersurfaces H and H' of the boundary with either $H, H' \subset \partial_{\mathrm{reg}}\overline{M}$ or $H, H' \subset \partial_{\mathrm{sing}}\overline{M}$ we either have $H \cap H' = \emptyset$ or $H = H'$. Consequently, the codimension two strata occur as intersections of regular and singular hypersurfaces only. Both $\partial_{\mathrm{reg}}\overline{M}$ and $\partial_{\mathrm{sing}}\overline{M}$ are smooth compact manifolds with boundary, and we have $\partial\big(\partial_{\mathrm{reg}}\overline{M}\big) = \partial\big(\partial_{\mathrm{sing}}\overline{M}\big) = \partial_{\mathrm{reg}}\overline{M} \cap \partial_{\mathrm{sing}}\overline{M}$.

Let $2\overline{M}_{\mathrm{reg}}$ be the double of $\overline{M}$ across the regular boundary hypersurfaces. $2\overline{M}_{\mathrm{reg}}$ is a compact smooth manifold with boundary, and we have $\overline{M} \subset 2\overline{M}_{\mathrm{reg}}$. Let r^+ be the restriction operator for distributions on the interior of $2\overline{M}_{\mathrm{reg}}$ to the interior of $\overline{M}$, and define as usual $H_b^s(\overline{M}) := r^+ H_b^s(2\overline{M}_{\mathrm{reg}})$ equipped with the quotient topology. More generally, if E is a (Hermitian) vector bundle on $\overline{M}$ and x is a defining function for $\partial_{\mathrm{sing}}\overline{M}$, we obtain the weighted space $x^\gamma H_b^s(\overline{M};E)$ in the way just described.

Corollary 3.4. *The statement of Theorem 3.1 is valid for the weighted H_b^s-spaces on compact manifolds with corners of codimension two.*

Proof. We just need to note that the embedding $x^\gamma H_b^s(\overline{M}) \hookrightarrow x^{\gamma'} H_b^{s'}(\overline{M})$ can be written as the composition of the maps $r^+ \circ \iota \circ e_{s,\gamma}$, where $e_{s,\gamma} : x^\gamma H_b^s(\overline{M}) \to x^\gamma H_b^s(2\overline{M}_{\mathrm{reg}})$ is an extension operator, $\iota : x^\gamma H_b^s(2\overline{M}_{\mathrm{reg}}) \to x^{\gamma'} H_b^{s'}(2\overline{M}_{\mathrm{reg}})$ is the embedding that belongs to $\mathfrak{S}_p$ according to Theorem 3.1 (provided that $\gamma > \gamma'$ and $s > s' + n/p$ as is assumed here), and $r^+ : x^{\gamma'} H_b^{s'}(2\overline{M}_{\mathrm{reg}}) \to x^{\gamma'} H_b^{s'}(\overline{M})$ is the restriction operator. □

4. Cone operators and rays of minimal growth

In this section we compile the definitions and some of the basic results about cone operators. For detailed accounts we refer to the monograph [17] and the papers [10, 11, 15]. Boundary value problems for cone operators are discussed in [5, 16]. There are many more references that could be mentioned, but those are the ones that are closest to our present scope since they emphasize the unbounded operator aspect and discuss operators of general form.

Since rays of minimal growth are essential in the context of the present paper, we will proceed to review the results from [10, 11, 12, 16] about when a ray $\Gamma \subset \mathbb{C}$ is a ray of minimal growth for a closed extension of an elliptic cone operator.

Let $\overline{M}$ be a smooth, compact n-manifold with boundary Y. The natural framework for cone geometry is the c-cotangent bundle

$$ {}^c\pi : {}^cT^*\overline{M} \to \overline{M}, \tag{4.1} $$

see [10], a vector bundle whose space of smooth sections is in one-to-one correspondence with the space of all smooth 1-forms on $\overline{M}$ that are conormal to Y, i.e., all $\omega \in C^\infty(\overline{M}, T^*\overline{M})$ whose pullback to Y vanishes. The isomorphism is given by a bundle homomorphism

$$ {}^c\mathrm{ev} : {}^cT^*\overline{M} \to T^*\overline{M} \tag{4.2} $$

which is an isomorphism over $\mathring{M}$. In coordinates $(x, y_1, \dots, y_{n-1})$ near the boundary, where x is a defining function for Y, a local frame for ${}^cT^*\overline{M}$ is given by the sections mapped by ${}^c\mathrm{ev}$ to the forms dx, $xdy_1, \dots, xdy_{n-1}$.

By a c-metric we mean any metric on the dual of ${}^cT^*\overline{M}$. Such a metric induces (via the homomorphism (4.2)) a Riemannian metric ${}^cg$ on $\mathring{M}$. In coordinates near the boundary as in the previous paragraph, ${}^cg$ is represented as a smooth symmetric 2-cotensor

$$ {}^cg = g_{00}\, dx \otimes dx + \sum_{j=1}^{n-1} g_{0j}\, dx \otimes xdy_j + \sum_{i=1}^{n-1} g_{i0}\, xdy_i \otimes dx + \sum_{i,j=1}^{n-1} g_{ij}\, xdy_i \otimes xdy_j; $$

the matrix (g_{ij}) depends smoothly on (x,y) and is positive definite up to $x = 0$.

Special cases of c-metrics are warped and straight cone metrics. A warped cone metric is a Riemannian metric on $\overline{M}$ such that there is a diffeomorphism of a neighborhood U of Y in M to $[0,\varepsilon)\times Y$ under which the metric takes on the form $dx^2+x^2g_Y(x)$ for a family of metrics $g_Y(x)$ on Y which is smooth up to $x=0$; here x is of course the variable in $[0,\varepsilon)$. If the diffeomorphism is such that $g_Y(x)$ is in fact independent of x for small ε, then ${}^cg$ is a straight cone metric.

Let $E,F\to\overline{M}$ be (Hermitian) vector bundles. A cone differential operator of order m acting from sections of E to sections of F is an element A of $x^{-m}\operatorname{Diff}^m_b(\overline{M};E,F)$, where $\operatorname{Diff}^m_b(\overline{M};E,F)$ is the space of totally characteristic differential operators of order m, see Section 3. Thus A is a linear differential operator $C^\infty(\overset{\circ}{M};E)\to C^\infty(\overset{\circ}{M};F)$, of order m, which near any point in Y, in coordinates $(x,y_1,\dots,y_{n-1})$ as above, is of the form

$$A=x^{-m}\sum_{k+|\alpha|\le m}a_{k\alpha}(x,y)(xD_x)^kD_y^\alpha \tag{4.3}$$

with coefficients $a_{k\alpha}$ smooth up to $x=0$. For example, the Laplacian with respect to any c-metric is a cone differential operator of order 2.

The standard principal symbol of a cone operator A over the interior determines, with the aid of the map ${}^c\mathrm{ev}$ in (4.2), a smooth homomorphism ${}^c\pi^*E\to{}^c\pi^*F$. This is the c-principal symbol ${}^c\boldsymbol{\sigma}(A)$ of A. In local coordinates near Y,

$${}^c\boldsymbol{\sigma}(A)=\sum_{k+|\alpha|=m}a_{k\alpha}(x,y)\xi^k\eta^\alpha.$$

The operator A is said to be c-elliptic if ${}^c\boldsymbol{\sigma}(A)$ is invertible on ${}^cT^*M\backslash 0$.

In the sequel we fix an operator $A\in x^{-m}\operatorname{Diff}^m_b(\overline{M};E)$, $m>0$, and assume that it is c-elliptic. For every weight $\gamma\in\mathbb{R}$ the operator A is a densely defined unbounded operator

$$A:C_c^\infty(\overset{\circ}{M};E)\subset x^\gamma L_b^2(\overline{M};E)\to x^\gamma L_b^2(\overline{M};E). \tag{4.4}$$

Observe that the geometric L^2-space with respect to any c-metric on $\overline{M}$ and Hermitian metric on E is the space $x^{-n/2}L^2_b(\overline{M};E)$, where $n=\dim\overline{M}$.

For any choice of $\gamma\in\mathbb{R}$ there are two canonical closed extensions of A:

$$\begin{aligned}\mathcal{D}_{\min}&=\text{domain of the closure of (4.4)},\\ \mathcal{D}_{\max}&=\{u\in x^\gamma L^2_b(\overline{M};E);\ Au\in x^\gamma L^2_b(\overline{M};E)\}.\end{aligned}$$

These are complete with respect to the graph norm, $\|u\|_A=\|u\|+\|Au\|$, and the former is a subspace of the latter. The following theorem lists basic results proved in [15, 17].

Theorem 4.5. *Let $A\in x^{-m}\operatorname{Diff}^m_b(\overline{M};E)$, $m>0$, be c-elliptic, and consider A an unbounded operator in $x^\gamma L^2_b(\overline{M};E)$ as described above.*

(a) $\dim\mathcal{D}_{\max}/\mathcal{D}_{\min}<\infty$. In particular, every intermediate space $\mathcal{D}_{\min}\subset\mathcal{D}\subset\mathcal{D}_{\max}$ gives rise to a closed extension

$$A_{\mathcal{D}}:\mathcal{D}\subset x^\gamma L^2_b(\overline{M};E)\to x^\gamma L^2_b(\overline{M};E).$$

(b) All closed extensions $A_{\mathcal{D}}$ of A are Fredholm. Moreover,

$$\operatorname{ind}A_{\mathcal{D}}=\operatorname{ind}A_{\mathcal{D}_{\min}}+\dim\mathcal{D}/\mathcal{D}_{\min}. \tag{4.6}$$

(c) $\mathcal{D}_{\min} = \bigcap_{\varepsilon>0} x^{\gamma+m-\varepsilon} H_b^m(\overline{M};E) \cap \mathcal{D}_{\max}$.

Moreover, $x^{\gamma+m} H_b^m(\overline{M};E) \subset \mathcal{D}_{\min}$*, and there is equality* $x^{\gamma+m} H_b^m(\overline{M};E) = \mathcal{D}_{\min}$ *if and only if* $\operatorname{spec}_b(A) \cap \{\sigma \in \mathbb{C};\ \Im(\sigma) = -\gamma - m\} = \emptyset$*. The set* $\operatorname{spec}_b(A) \subset \mathbb{C}$ *is the boundary spectrum of* A*, see* [19]*, a discrete set that contains at most finitely many points in each horizontal strip of finite width.*

(d) *There exists* $\varepsilon > 0$ *such that* $\mathcal{D}_{\max} \hookrightarrow x^{\gamma+\varepsilon} H_b^m(\overline{M};E)$.

By (d) of Theorem 4.5 and Theorem 3.1 we immediately obtain the following corollary.

Corollary 4.7. *Under the assumptions of Theorem 4.5 the embedding* $\mathcal{D}_{\max} \hookrightarrow x^{\gamma} L_b^2(\overline{M};E)$ *belongs to* $\mathfrak{S}^+_{n/m}$*, see* (2.2).

In particular, the embeddings of the domains $\mathcal{D} \hookrightarrow x^{\gamma} L_b^2(\overline{M};E)$ *of all closed extensions* $A_{\mathcal{D}}$ *of* A *in* $x^{\gamma} L_b^2(\overline{M};E)$ *belong to* $\mathfrak{S}^+_{n/m}$.

In the study of rays of minimal growth for closed extensions of A the normal operator $A_\wedge$ associated with A plays a significant role. $A_\wedge$ is an operator acting in sections on the inward pointing half of the normal bundle of Y in $\overline{M}$. More precisely, $A_\wedge$ is defined as follows:

We first note that any choice of defining function x for Y trivializes the normal bundle NY to $Y \times \mathbb{R}$. x induces the map $x_\wedge = dx$ on NY, and the trivialization $NY \cong Y \times \mathbb{R}$ then is such that $x_\wedge$ corresponds to the projection on the second coordinate on $Y \times \mathbb{R}$. To simplify notation, we will just write x for $x_\wedge$ from now on. Let $Y^\wedge = Y \times \overline{\mathbb{R}}_+$ be the inward pointing half of the normal bundle. The bundle $E|_Y$ lifts to $Y^\wedge$ and carries a natural Hermitian metric and connection induced by the metric and connection given on E. As is custom in the literature on cone operators, this bundle on $Y^\wedge$ is for sake of simplicity also denoted by E. On $Y^\wedge$ we consider the b-density $\frac{dx}{x} \otimes \mathfrak{m}_Y$ with a fixed density $\mathfrak{m}_Y$ on Y (lifted to $Y^\wedge$).

Choose a collar neighborhood U of Y in $\overline{M}$. Pull-back and parallel transport induce an isomorphism $x^{\gamma} L_b^2(U;E) \cong x^{\gamma} L_b^2(Y \times [0,\varepsilon);E)$ for $\varepsilon > 0$ small enough (choosing the defining function x and the collar neighborhood U properly even produces a unitary map, but this will not be essential for us here). Hence, locally near Y, L^2-sections of E on $\overline{M}$ can be identified with L^2-sections of E on $Y^\wedge$. This identification extends to distributional sections and restricts to smooth sections (with compact support). So, if $A \in x^{-m} \operatorname{Diff}_b^m(\overline{M};E)$, then near Y we can now write

$$A = x^{-m} \sum_{k=0}^{m} a_k(x)(xD_x)^k \tag{4.8}$$

with $a_k \in C^\infty([0,\varepsilon), \operatorname{Diff}^{m-k}(Y;E|_Y))$. The normal operator associated with A is the operator

$$A_\wedge = x^{-m} \sum_{k=0}^{m} a_k(0)(xD_x)^k : C_c^\infty(\mathring{Y}^\wedge;E) \to C^\infty(\mathring{Y}^\wedge;E). \tag{4.9}$$

For every $\gamma \in \mathbb{R}$, $A_\wedge$ is an unbounded operator

$$A_\wedge : C_c^\infty(\mathring{Y}^\wedge;E) \subset x^{\gamma} L_b^2(Y^\wedge;E) \to x^{\gamma} L_b^2(Y^\wedge;E).$$

Like A, $A_\wedge$ has the canonical closed minimal and maximal extensions $\mathcal{D}_{\wedge,\min}$ and $\mathcal{D}_{\wedge,\max}$. There is a natural isomorphism

$$\theta : \mathcal{D}_{\max}/\mathcal{D}_{\min} \to \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min} \tag{4.10}$$

constructed in [10, 11] (and subsequently reviewed in [12, 13]). Without going into further technical details, we just note that the construction of θ follows a simple algorithm of m steps, where m is the order of A. It involves the first m Taylor coefficients of the expansions of the $a_k(x)$ in (4.8) (that is to say the conormal symbols of the operator A up to order m). In the special case that A has constant coefficients, i.e. the $a_k(x)$ are independent of x for small x, we simply have $\theta\big(u+\mathcal{D}_{\min}\big)=\omega u+\mathcal{D}_{\wedge,\min}$, where $\omega\in C_c^\infty([0,\varepsilon))$ is a cut-off function near $x=0$ that we consider a function on $\overline{M}$ supported near Y (this representation of θ involves passage for functions on $\overline{M}$ supported near Y to functions on $Y^\wedge$ as was discussed earlier).

Using (4.10) we can associate with any domain $\mathcal{D}_{\min}\subset\mathcal{D}\subset\mathcal{D}_{\max}$ for A a domain $\mathcal{D}_{\wedge,\min}\subset\mathcal{D}_\wedge\subset\mathcal{D}_{\wedge,\max}$ for $A_\wedge$ via

$$\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}=\theta\big(\mathcal{D}/\mathcal{D}_{\min}\big). \tag{4.11}$$

Now let $\Gamma=\{re^{i\theta};\ r\geq 0\}\subset\mathbb{C}$ be a ray. The following theorem, proved in [11], gives verifiable criteria for Γ to be a ray of minimal growth for the closed extension $A_{\mathcal{D}}$ of an elliptic cone operator.

Theorem 4.12. *Let $A\in x^{-m}\operatorname{Diff}_b^m(\overline{M};E)$, $m>0$, be c-elliptic, and let $A_{\mathcal{D}}$ be any closed extension of A in $x^\gamma L_b^2(\overline{M};E)$. We assume that*

- *A is c-elliptic with parameter in Γ, i.e. the c-principal symbol ${}^c\boldsymbol{\sigma}(A)$ does not have spectrum in Γ;*
- *Γ is a ray of minimal growth for the closed extension $A_{\wedge,\mathcal{D}_\wedge}$ of the normal operator $A_\wedge$ in $x^\gamma L_b^2(Y^\wedge;E)$, where $\mathcal{D}_\wedge$ is the associated domain to $\mathcal{D}$ according to* (4.11).

Then Γ is a ray of minimal growth for $A_{\mathcal{D}}$.

The second assumption on the normal operator can be phrased conveniently in geometric terms that involve the action

$$\kappa_\varrho u(x,y)=u(\varrho x,y),\,\varrho>0, \tag{4.13}$$

that is defined for functions on $Y^\wedge$ (for sections of bundles the definition of this action involves in addition parallel transport in the fibres). Since both $\mathcal{D}_{\wedge,\max}$ and $\mathcal{D}_{\wedge,\min}$ are invariant with respect to this action, it descends to an action on the quotient $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ and therefore induces flows on the various Grassmannians of its subspaces.

For any domain $\mathcal{D}_\wedge$ let $\Omega^-(\mathcal{D}_\wedge)$ consist of all domains $\tilde{\mathcal{D}}_\wedge$ of closed extensions of $A_\wedge$ such that $\dim\tilde{\mathcal{D}}_\wedge/\mathcal{D}_{\wedge,\min}=\dim\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}$, so these quotient spaces belong to the same Grassmannian, and such that there exists a sequence $\varrho_k\to 0$ such that

$$\kappa_{\varrho_k}\big(\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}\big)\to\tilde{\mathcal{D}}_\wedge/\mathcal{D}_{\wedge,\min}\text{ as }k\to\infty$$

in that Grassmannian. It was shown in [14] that $\Omega^-(\mathcal{D}_\wedge)$ has topologically the structure of an embedded torus.

Now, provided that A is c-elliptic with parameter in Γ, it was proved in [10, 12] that Γ is a ray of minimal growth for $A_\wedge$ with domain $\mathcal{D}_\wedge$ if and only if, for some $\lambda_0\in\Gamma$, $A_\wedge-\lambda_0:\tilde{\mathcal{D}}_\wedge\to x^\gamma L_b^2(Y^\wedge;E)$ is invertible for all $\tilde{\mathcal{D}}_\wedge\in\Omega^-(\mathcal{D}_\wedge)$.

Let us now proceed with the corresponding discussion for realizations of elliptic boundary value problems on manifolds with conical singularities.

Let $\overline{M}$ be a compact n-manifold with corners of codimension two, and let $\partial\overline{M} = \partial_{\mathrm{reg}}\overline{M} \cup \partial_{\mathrm{sing}}\overline{M}$, where both $\partial_{\mathrm{reg}}\overline{M}$ and $\partial_{\mathrm{sing}}\overline{M}$ are smooth manifolds with boundary as described in Section 3 before Corollary 3.4. Let $2\overline{M}_{\mathrm{reg}}$ be the double of $\overline{M}$ across $\partial_{\mathrm{reg}}\overline{M}$. $2\overline{M}_{\mathrm{reg}}$ is a smooth compact manifold with boundary $2\partial_{\mathrm{sing}}\overline{M}$, the double of $\partial_{\mathrm{sing}}\overline{M}$ across its boundary.

We obtain the relevant objects on $\overline{M}$ by restriction of the corresponding objects from $2\overline{M}_{\mathrm{reg}}$. For example, the c-cotangent bundle ${}^cT^*\overline{M}$ is by definition ${}^cT^*2\overline{M}_{\mathrm{reg}}|_{\overline{M}}$. Similarly, we consider (Hermitian) vector bundles $E \to \overline{M}$ that are restrictions of (Hermitian) vector bundles from $2\overline{M}_{\mathrm{reg}}$. Let x be a defining function for $2\partial_{\mathrm{sing}}\overline{M}$ in $2\overline{M}_{\mathrm{reg}}$. For any $m \in \mathbb{N}_0$ and vector bundles E and F, let $x^{-m}\operatorname{Diff}_b^m(\overline{M}; E, F)$ be the space of cone differential operators of order m on $\overline{M}$ acting from sections of the bundle E to sections of F. Every operator in this space is obtained by restricting a corresponding cone differential operator from the double $2\overline{M}_{\mathrm{reg}}$ to $\overline{M}$.

In the sequel, we fix an operator $A \in x^{-m}\operatorname{Diff}_b^m(\overline{M}; E)$, $m > 0$, and a collection of operators $B_j \in x^{-m_j}\operatorname{Diff}_b^{m_j}(\overline{M}; E, F_j)$, $m_j < m$, $j = 1, \ldots, N$, and consider the following boundary value problem with spectral parameter $\lambda \in \Gamma = \{re^{i\theta};\ r \geq 0\} \subset \mathbb{C}$:

$$\left.\begin{aligned} (A-\lambda)u = f \text{ in } \mathring{\overline{M}}, \\ Tu = \begin{pmatrix} \mathfrak{r}_{\partial_{\mathrm{reg}}\overline{M}} \circ B_1 u \\ \vdots \\ \mathfrak{r}_{\partial_{\mathrm{reg}}\overline{M}} \circ B_N u \end{pmatrix} = 0 \text{ on } \partial_{\mathrm{reg}}\overline{M}, \end{aligned}\right\} \tag{4.14}$$

where $\mathfrak{r}_{\partial_{\mathrm{reg}}\overline{M}} : v \mapsto v|_{\partial_{\mathrm{reg}}\overline{M}}$ is the trace operator. More precisely, for any given weight $\gamma \in \mathbb{R}$, we consider the spectral problem for the operator A_T in $x^\gamma L_b^2(\overline{M}; E)$ that acts like A with domain $\mathcal{D}(A_T)$, where $\mathcal{D}(A_T)$ is any intermediate space $\mathcal{D}_{\min}(A_T) \subset \mathcal{D}(A_T) \subset \mathcal{D}_{\max}(A_T)$, and

$$\begin{aligned} \mathcal{D}_{\max}(A_T) &= \{u \in x^\gamma H_b^m(\overline{M}; E);\ Au \in x^\gamma L_b^2(\overline{M}; E) \text{ and } Tu = 0 \text{ on } \partial_{\mathrm{reg}}\overline{M}\}, \\ \mathcal{D}_{\min}(A_T) &= \mathcal{D}_{\max}(A_T) \cap \bigcap_{\varepsilon > 0} x^{\gamma+m-\varepsilon} H_b^m(\overline{M}; E). \end{aligned}$$

We henceforth assume that

$$\left.\begin{minipage}{0.8\textwidth}
\begin{itemize}
\item A is c-elliptic with parameter in Γ, i.e., ${}^c\boldsymbol{\sigma}(A) - \lambda$ is invertible everywhere on $\big({}^cT^*\overline{M} \times \Gamma\big) \setminus 0$;
\item the c-principal boundary symbol with parameter
$$\begin{pmatrix} {}^c\boldsymbol{\sigma}_\partial(A) - \lambda \\ {}^c\boldsymbol{\sigma}_\partial(T) \end{pmatrix} : {}^c\mathscr{S}_+ \otimes {}^c\pi^*E|_{\partial_{\mathrm{reg}}\overline{M}} \to \begin{matrix} {}^c\mathscr{S}_+ \otimes {}^c\pi^*E|_{\partial_{\mathrm{reg}}\overline{M}} \\ \oplus \\ \bigoplus_{j=1}^N {}^c\pi^*F_j|_{\partial_{\mathrm{reg}}\overline{M}} \end{matrix}$$
is invertible on $\big({}^cT^*\partial_{\mathrm{reg}}\overline{M} \times \Gamma\big) \setminus 0$, where ${}^c\pi : {}^cT^*\partial_{\mathrm{reg}}\overline{M} \to \partial_{\mathrm{reg}}\overline{M}$ is the canonical projection.
\end{itemize}
\end{minipage}\right\} \tag{4.15}$$

We proceed to explain the notion of c-principal boundary symbol from (4.15). Let y_1 be a defining function for $\partial_{\mathrm{reg}}\overline{M}$ such that $dx \wedge dy_1 \neq 0$ on $\partial_{\mathrm{reg}}\overline{M} \cap \partial_{\mathrm{sing}}\overline{M}$. In

local coordinates near $\partial_{\mathrm{reg}}\overline{M}$ write

$$ {}^c\boldsymbol{\sigma}(A) = \sum_{j+|\alpha|=m} a_{j,\alpha}(y',y_1)\eta'^{\alpha}\eta_1^j. $$

Then the c-principal boundary symbol of A is

$$ {}^c\boldsymbol{\sigma}_\partial(A) = \sum_{j+|\alpha|=m} a_{j,\alpha}(y',0)\eta'^{\alpha}D_{y_1}^j : \mathscr{S}(\overline{\mathbb{R}}_+)\otimes\mathbb{C}^K \to \mathscr{S}(\overline{\mathbb{R}}_+)\otimes\mathbb{C}^K, $$

where $K=\dim E$. Globally this leads to

$$ {}^c\boldsymbol{\sigma}_\partial(A) : {}^c\mathscr{S}_+ \otimes {}^c\pi^*E|_{\partial_{\mathrm{reg}}\overline{M}} \to {}^c\mathscr{S}_+ \otimes {}^c\pi^*E|_{\partial_{\mathrm{reg}}\overline{M}}, $$

where ${}^c\mathscr{S}_+ \to {}^cT^*\partial_{\mathrm{reg}}\overline{M}$ is a vector bundle with fiber $\mathscr{S}(\overline{\mathbb{R}}_+)$.

Analogously, we have

$$ {}^c\boldsymbol{\sigma}_\partial(B_j) : {}^c\mathscr{S}_+ \otimes {}^c\pi^*E|_{\partial_{\mathrm{reg}}\overline{M}} \to {}^c\mathscr{S}_+ \otimes {}^c\pi^*F_j|_{\partial_{\mathrm{reg}}\overline{M}}, \quad j=1,\dots,N. $$

Let ${}^c\boldsymbol{\sigma}_\partial(\mathfrak{r}_{\partial_{\mathrm{reg}}\overline{M}}) : u \to u(0)$ fiberwise in ${}^c\mathscr{S}_+$. Combined this gives

$$ {}^c\boldsymbol{\sigma}_\partial(T) = \begin{pmatrix} {}^c\boldsymbol{\sigma}_\partial(\mathfrak{r}_{\partial_{\mathrm{reg}}\overline{M}}) \circ {}^c\boldsymbol{\sigma}_\partial(B_1) \\ \vdots \\ {}^c\boldsymbol{\sigma}_\partial(\mathfrak{r}_{\partial_{\mathrm{reg}}\overline{M}}) \circ {}^c\boldsymbol{\sigma}_\partial(B_N) \end{pmatrix} : {}^c\mathscr{S}_+ \otimes {}^c\pi^*E|_{\partial_{\mathrm{reg}}\overline{M}} \to \bigoplus_{j=1}^N {}^c\pi^*F_j|_{\partial_{\mathrm{reg}}\overline{M}}, $$

the c-principal boundary symbol of the boundary condition T.

For the discussion of rays of minimal growth, we also need to impose a parameter-dependent ellipticity condition that is associated with $\overline{Y}=\partial_{\mathrm{sing}}\overline{M}$. This condition involves the normal operator $A_\wedge$ of A and a corresponding normal boundary value problem $T_\wedge$ for $A_\wedge$ on $\overline{Y}^\wedge$. The definition of $A_\wedge$ is exactly like in (4.9). Likewise, there are normal operators $B_{j,\wedge}$ associated with the operators B_j, $j=1,\dots,N$. The normal operator associated with T is then

$$ T_\wedge = \begin{pmatrix} \mathfrak{r}_{(\partial\overline{Y})^\wedge} \circ B_{1,\wedge} \\ \vdots \\ \mathfrak{r}_{(\partial\overline{Y})^\wedge} \circ B_{N,\wedge} \end{pmatrix} : C_c^\infty(\overline{Y}^\wedge;E) \to C_c^\infty((\partial\overline{Y})^\wedge;\bigoplus_{j=1}^N F_j). $$

For the previously fixed weight $\gamma\in\mathbb{R}$ we consider the spectral problem for the realizations of $A_\wedge$ subject to $T_\wedge u=0$ in $x^\gamma L_b^2(\overline{Y}^\wedge;E)$. More precisely, we consider the operator $A_{\wedge,T_\wedge}$ that acts like $A_\wedge$ with domain $\mathcal{D}_\wedge(A_{\wedge,T_\wedge})$, where $\mathcal{D}_\wedge(A_{\wedge,T_\wedge})$ is any intermediate space $\mathcal{D}_{\wedge,\min}(A_{\wedge,T_\wedge}) \subset \mathcal{D}_\wedge(A_{\wedge,T_\wedge}) \subset \mathcal{D}_{\wedge,\max}(A_{\wedge,T_\wedge})$. Here

$$ \begin{aligned} \mathcal{D}_{\wedge,\max}(A_{\wedge,T_\wedge}) &= \{u\in\mathcal{K}^{m,\gamma}(\overline{Y}^\wedge;E)_\gamma;\ A_\wedge u \in x^\gamma L_b^2(\overline{Y}^\wedge;E) \text{ and } T_\wedge u = 0\}, \\ \mathcal{D}_{\wedge,\min}(A_{\wedge,T_\wedge}) &= \mathcal{D}_{\wedge,\max}(A_{\wedge,T_\wedge}) \cap \bigcap_{\varepsilon>0} \mathcal{K}^{m,\gamma+m-\varepsilon}(\overline{Y}^\wedge;E)_\gamma, \end{aligned} $$

and for $s,\delta,\delta'\in\mathbb{R}$,

$$ \mathcal{K}^{s,\delta}(\overline{Y}^\wedge)_{\delta'} = \omega x^\delta H_b^s(\overline{Y}^\wedge) + (1-\omega)x^{\delta'+n/2}H_{\mathrm{cone}}^s(\overline{Y}^\wedge) $$

is a weighted cone Sobolev space on $\overline{Y}^\wedge$, see [20, 8, 16]. Here $\omega\in C_c^\infty(\overline{\mathbb{R}}_+)$ is a cut-off function near zero.

It was shown in [16] that under our present assumptions (4.15) there exists a natural isomorphism

$$ \theta : \mathcal{D}_{\max}(A_T)/\mathcal{D}_{\min}(A_T) \to \mathcal{D}_{\wedge,\max}(A_{\wedge,T_\wedge})/\mathcal{D}_{\wedge,\min}(A_{\wedge,T_\wedge}) $$

similar to (4.10) that allows passage from domains $\mathcal{D}(A_T)$ of realizations of A subject to $Tu=0$ to associated domains $\mathcal{D}_\wedge(A_{\wedge,T_\wedge})$ of realizations of $A_\wedge$ subject to $T_\wedge u = 0$ via

$$\mathcal{D}_\wedge(A_{\wedge,T_\wedge})/\mathcal{D}_{\wedge,\min}(A_{\wedge,T_\wedge}) = \theta\big(\mathcal{D}(A_T)/\mathcal{D}_{\min}(A_T)\big), \tag{4.16}$$

see also (4.11). Moreover, the quotient spaces $\mathcal{D}_{\max}(A_T)/\mathcal{D}_{\min}(A_T)$ and correspondingly $\mathcal{D}_{\wedge,\max}(A_{\wedge,T_\wedge})/\mathcal{D}_{\wedge,\min}(A_{\wedge,T_\wedge})$ are finite dimensional.

In addition to (4.15) we will require the following parameter-dependent ellipticity condition associated with $\partial_{\mathrm{sing}}\overline{M}$:

The ray Γ is a ray of minimal growth for $A_{\wedge,T_\wedge}$ with the associated domain $\mathcal{D}_\wedge(A_{\wedge,T_\wedge})$ to $\mathcal{D}(A_T)$ according to (4.16). (4.17)

The following theorem is the main result of [16].

Theorem 4.18. *Consider the realization A_T of A subject to the boundary condition $Tu=0$ on $\overline{M}$ in $x^\gamma L^2_b(\overline{M};E)$ with domain $\mathcal{D}(A_T)$, where $\mathcal{D}_{\min}(A_T)\subset\mathcal{D}(A_T)\subset\mathcal{D}_{\max}(A_T)$, and let $\Gamma=\{re^{i\theta};\ r\geq 0\}\subset\mathbb{C}$ be a ray. Assume that the parameter-dependent ellipticity conditions* (4.15) *and* (4.17) *are fulfilled.*

Then Γ is a ray of minimal growth for the operator $A_T:\mathcal{D}(A_T)\to x^\gamma L^2_b(\overline{M};E)$.

Under the assumptions of Theorem 4.18 it was shown in [16] that *all* realizations of A_T with domains between $\mathcal{D}_{\min}(A_T)$ and $\mathcal{D}_{\max}(A_T)$ are closed operators in the functional analytic sense, that they are all Fredholm, and, moreover, that $\mathcal{D}_{\max}(A_T)\hookrightarrow x^{\gamma+\varepsilon}H^m_b(\overline{M};E)$ for a sufficiently small $\varepsilon>0$. In view of Corollary 3.4, the latter implies the following.

Corollary 4.19. *Under the assumptions of Theorem 4.18, the embedding of the domain $\mathcal{D}(A_T)\hookrightarrow x^\gamma L^2_b(\overline{M};E)$ belongs to $\mathfrak{S}^+_{n/m}$, see* (2.2).

Finally, we note that the assumption (4.17) can be checked effectively using the dilation group κ_ϱ from (4.13) and the induced flow on the Grassmannians of subspaces of the quotient $\mathcal{D}_{\wedge,\max}(A_{\wedge,T_\wedge})/\mathcal{D}_{\wedge,\min}(A_{\wedge,T_\wedge})$ analogously to the case of closed extensions of cone operators without boundary conditions, see the explanation after Theorem 4.12.

5. Main theorems and examples

What remains to be done is to combine the results from the previous sections to obtain our main Theorems 5.1 and 5.3 about the completeness of generalized eigenfunctions for elliptic cone operators.

Theorem 5.1. *Let $\overline{M}$ be a smooth compact n-manifold with boundary Y, and let $A\in x^{-m}\operatorname{Diff}^m_b(\overline{M};E)$, $m>0$, be c-elliptic. Fix a weight $\gamma\in\mathbb{R}$, and consider the closed extension*

$$A_{\mathcal{D}}:\mathcal{D}\subset x^\gamma L^2_b(\overline{M};E)\to x^\gamma L^2_b(\overline{M};E)$$

of A. We assume that there are rays

$$\Gamma_j=\{re^{i\theta_j};\ r\geq 0\},\quad j=1,\ldots,J,$$

in the complex plane such that all angles enclosed by any two adjacent rays are $\leq\frac{\pi m}{n}$, and such that for any such ray Γ,

- *${}^c\boldsymbol{\sigma}(A)-\lambda$ is invertible on $\big({}^cT^*\overline{M}\times\Gamma\big)\setminus 0$;*

- Γ *is a ray of minimal growth for*

$$A_\wedge : \mathcal{D}_\wedge \subset x^\gamma L^2_b(Y^\wedge; E) \to x^\gamma L^2_b(Y^\wedge; E)$$

for the associated domain $\mathcal{D}_\wedge$ *to* $\mathcal{D}$ *according to* (4.11).

Then the system of generalized eigenfunctions of $A_{\mathcal{D}}$ *is complete in* $x^\gamma L^2_b(\overline{M}; E)$.

As was pointed out after Theorem 4.12, we note that the assumption that Γ be a ray of minimal growth for $A_\wedge$ with domain $\mathcal{D}_\wedge$ can be checked effectively using the dilation group κ_ϱ from (4.13) and the induced flow on the Grassmannian of subspaces of the quotient $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ that contains the subspace $\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}$. We will illustrate this in Example 5.2 below.

In the special case where $\mathcal{D} = \mathcal{D}_{\min} = x^{\gamma+m} H^m_b(\overline{M}; E)$, Theorem 5.1 was proved by Egorov, Kondratiev, and Schulze in [7]. The following example illustrates why the general result is relevant. Further information pertaining to this example can be found in [15] (as far as the Friedrichs domain is concerned), and, in particular, in [12].

Example 5.2. Let $\overline{M}$ be a smooth compact 2-manifold with boundary $Y = \mathbb{S}^1$. Fix a collar neighborhood map $U \cong Y \times [0, \varepsilon)$ of the boundary, and a defining function x for Y that coincides in $\overset{\circ}{U}$ with the projection to the coordinate in $[0, \varepsilon)$. Let ${}^c g$ be a Riemannian metric on $\overline{M}$ that in the splitting of variables $(y, x) \in Y \times [0, \varepsilon)$ near the boundary takes the form ${}^c g = dx^2 + x^2 g_Y(x)$ for a smooth family of metrics $g_Y(x)$ on Y up to $x = 0$, and assume that $g_Y(0)$ is the standard round metric on $\mathbb{S}^1$.

${}^c g$ is a special c-metric as was discussed at the beginning of Section 4, and the positive Laplacian $\Delta = \Delta_{{}^c g} \in x^{-2} \operatorname{Diff}^2_b(\overline{M})$ is a cone differential operator. Its c-principal symbol ${}^c\boldsymbol{\sigma}(\Delta)$ is the metric induced by ${}^c g$ on ${}^cT^*\overline{M}$. Consequently, ${}^c\boldsymbol{\sigma}(\Delta) - \lambda$ is invertible for all $\lambda \notin \overline{\mathbb{R}}_+$, i.e., Δ is c-elliptic with parameter $\lambda \in \Gamma$ for *all* rays $\Gamma \neq \overline{\mathbb{R}}_+$.

The geometric L^2-space with respect to the metric ${}^c g$ is the space $x^{-1} L^2_b(\overline{M})$, and we consider Δ an unbounded operator

$$\Delta : C^\infty_c(\overset{\circ}{\overline{M}}) \subset x^{-1} L^2_b(\overline{M}) \to x^{-1} L^2_b(\overline{M}).$$

Δ has infinitely many selfadjoint and infinitely many nonselfadjoint closed extensions. In fact, $\dim \mathcal{D}_{\max}/\mathcal{D}_{\min} = 2$, $\operatorname{ind} \Delta_{\min} = -1$, and $\operatorname{ind} \Delta_{\max} = 1$. The domains $\mathcal{D}$ of closed extensions of Δ such that $\operatorname{ind} \Delta_{\mathcal{D}} = 0$ are the ones with $\dim \mathcal{D}/\mathcal{D}_{\min} = 1$. Using Theorem 5.1, we will proceed to argue that the system of generalized eigenfunctions of $\Delta_{\mathcal{D}}$ is complete in $x^{-1} L^2_b(\overline{M})$ for all domains $\mathcal{D}$ with $\dim \mathcal{D}/\mathcal{D}_{\min} = 1$. In particular, this includes all selfadjoint extensions (where the statement is trivial in view of the spectral theorem), but also infinitely many more nonselfadjoint extensions of Δ.

The normal operator $\Delta_\wedge$ associated to $\Delta_{{}^c g}$ on $Y^\wedge = \mathbb{S}^1 \times \overline{\mathbb{R}}_+$ is the positive Laplacian with respect to the metric $dx^2 + x^2 g_Y(0)$. In other words, it is the standard positive Laplacian in $\mathbb{R}^2 \setminus \{0\}$ in polar coordinates. Correspondingly, the space $x^{-1} L^2_b(Y^\wedge)$ is just the standard L^2-space on $\mathbb{R}^2 \setminus \{0\}$ with respect to Lebesgue measure, written in polar coordinates. We have

$$\mathcal{D}_{\wedge,\max} = \mathcal{D}_{\wedge,\min} \oplus \operatorname{span}\{\omega, \omega \log x\},$$

where $\omega \in C_c^\infty(\overline{\mathbb{R}}_+)$ is a cut-off function near zero. This gives an isomorphism

$$\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min} \cong \operatorname{span}\{1, \log x\},$$

and the action κ_ϱ from (4.13) that is induced on $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ is given by $\kappa_\varrho 1 = 1$ and $\kappa_\varrho \log x = \log(\varrho)\cdot 1 + \log x$ on the basis elements under this isomorphism.

Now let $\mathcal{D}_\wedge$ be any domain for $\Delta_\wedge$ with $\dim \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} = 1$. Then $\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}$ corresponds to $\operatorname{span}\{a\cdot 1 + b\cdot \log x\}$ for some $(a,b) \neq (0,0)$. κ_ϱ induces a flow on the Grassmannian of all subspaces $\tilde{\mathcal{D}}_\wedge/\mathcal{D}_{\wedge,\min}$ of $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ with $\dim \tilde{\mathcal{D}}_\wedge/\mathcal{D}_{\wedge,\min} = 1$. In that Grassmannian we have with the obvious identifications as $\varrho \to 0$

$$\begin{aligned}\kappa_\varrho\big(\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}\big) &= \operatorname{span}\{(a + b\log(\varrho))\cdot 1 + b\cdot \log x\} \\ = \operatorname{span}\{1 + \frac{b}{a+b\log(\varrho)}\cdot \log x\} &\underset{\varrho\to 0}{\longrightarrow} \operatorname{span}\{1\} = \mathcal{D}_{\wedge,F}/\mathcal{D}_{\wedge,\min},\end{aligned}$$

where $\mathcal{D}_{\wedge,F}$ is the domain of the Friedrichs extension of $\Delta_\wedge$. This shows that $\Omega^-(\mathcal{D}_\wedge) = \{\mathcal{D}_{\wedge,F}\}$ for any domain $\mathcal{D}_\wedge$ with $\dim \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} = 1$. Because $\Delta_\wedge - \lambda : \mathcal{D}_{\wedge,F} \to x^{-1}L_b^2(Y^\wedge)$ is invertible for all $\lambda \notin \overline{\mathbb{R}}_+$, we conclude that *all* rays $\Gamma \neq \overline{\mathbb{R}}_+$ are rays of minimal growth for *all* extensions of $\Delta_\wedge$ with domains $\mathcal{D}_\wedge$ such that $\dim \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} = 1$.

The arguments above now show that Theorem 5.1 is applicable for all closed extensions $\Delta_\mathcal{D}$ in $x^{-1}L_b^2(\overline{M})$ for *all* domains $\mathcal{D}_{\min} \subset \mathcal{D} \subset \mathcal{D}_{\max}$ with $\dim \mathcal{D}/\mathcal{D}_{\min} = 1$. Hence the system of generalized eigenfunctions of $\Delta_\mathcal{D}$ is complete in $x^{-1}L_b^2(\overline{M})$ for all these extensions.

This example is clearly not covered by [7]: Because $\operatorname{ind}\Delta_{\min} = -1$, the minimal extension of the Laplacian does not admit any rays of minimal growth. Likewise, $\Delta_{\wedge,\min}$ does not admit any rays of minimal growth. Moreover, in this example we also have $x^1H_b^2(\overline{M}) \subsetneq \mathcal{D}_{\min}$ (and the former is of infinite codimension in the latter), which shows that the scale of weighted b-Sobolev spaces that is widely used in the literature on cone operators cannot be expected to fit into the natural functional analytic framework of domains of closed extensions.

Theorem 5.3. *Let $\overline{M}$ be a compact n-manifold with corners of codimension two, $\partial\overline{M} = \partial_{\mathrm{reg}}\overline{M} \cup \partial_{\mathrm{sing}}\overline{M}$. Let x be a defining function for $\overline{Y} = \partial_{\mathrm{sing}}\overline{M}$, and let $A \in x^{-m}\operatorname{Diff}_b^m(\overline{M};E)$, $m > 0$. Let T be a vector of boundary conditions for A associated with $\partial_{\mathrm{reg}}\overline{M}$. Fix a weight $\gamma \in \mathbb{R}$, and consider the realization*

$$A_{T,\mathcal{D}} : \mathcal{D} \subset x^\gamma L_b^2(\overline{M};E) \to x^\gamma L_b^2(\overline{M};E)$$

of A subject to $Tu = 0$ on $\partial_{\mathrm{reg}}\overline{M}$ with domain $\mathcal{D}_{\min}(A_T) \subset \mathcal{D} \subset \mathcal{D}_{\max}(A_T)$. We assume that there are rays

$$\Gamma_j = \{re^{i\theta_j};\ r \geq 0\}, \quad j = 1,\dots,J,$$

in the complex plane such that all angles enclosed by any two adjacent rays are $\leq \frac{\pi m}{n}$, and such that for any such ray Γ,

- ${}^c\boldsymbol{\sigma}(A) - \lambda$ *is invertible on* $\big({}^cT^*\overline{M} \times \Gamma\big) \setminus 0$*;*
- $$\begin{pmatrix} {}^c\boldsymbol{\sigma}_\partial(A) - \lambda \\ {}^c\boldsymbol{\sigma}_\partial(T)\end{pmatrix} : {}^c\mathscr{S}_+ \otimes {}^c\pi^*E|_{\partial_{\mathrm{reg}}\overline{M}} \to \begin{matrix} {}^c\mathscr{S}_+ \otimes {}^c\pi^*E|_{\partial_{\mathrm{reg}}\overline{M}} \\ \oplus \\ \bigoplus_{j=1}^N {}^c\pi^*F_j|_{\partial_{\mathrm{reg}}\overline{M}}\end{matrix}$$

is invertible on $\big({}^cT^*\partial_{\mathrm{reg}}\overline{M}\times\Gamma\big)\setminus 0$*, where* ${}^c\pi : {}^cT^*\partial_{\mathrm{reg}}\overline{M}\to\partial_{\mathrm{reg}}\overline{M}$ *is the canonical projection;*

- Γ *is a ray of minimal growth for the realization*

$$A_{\wedge,T_\wedge} : \mathcal{D}_\wedge\subset x^\gamma L^2_b(\overline{Y}^\wedge;E)\to x^\gamma L^2_b(\overline{Y}^\wedge;E)$$

  of $A_\wedge$ *subject to* $T_\wedge u=0$ *with the associated domain* $\mathcal{D}_\wedge$ *to* $\mathcal{D}$ *according to* (4.16)*.*

Then the system of generalized eigenfunctions of $A_{T,\mathcal{D}}$ *is complete in* $x^\gamma L^2_b(\overline{M};E)$*.*

In the special case where

$$\mathcal{D}=\mathcal{D}_{\min}(A_T)=\{u\in x^{\gamma+m}H^m_b(\overline{M};E);\ Tu=0\},$$

Theorem 5.3 was obtained by Egorov, Kondratiev, and Schulze in [8]. The following example illustrates why the result is relevant in the general case.

Example 5.4. Let $\overline{\Omega}\subset\mathbb{R}^2$ be a bounded domain. We assume that $\partial\overline{\Omega}\setminus\{0\}$ is C^∞, and that the point 0 is an angular singularity. More specifically, after rotation, we assume that there is an angular domain $V=\{z\in\mathbb{C};\ z=xe^{i\theta},\ x\geq 0,\ 0\leq\theta\leq\alpha\}$, where $0<\alpha<2\pi$, such that there exists an $\varepsilon>0$ with $B_\varepsilon(0)\cap\overline{\Omega}=B_\varepsilon(0)\cap V$.

In $\overline{\Omega}$ we consider the positive Laplacian $\Delta=D^2_{x_1}+D^2_{x_2}$ subject to homogeneous Dirichlet boundary conditions on $\partial\overline{\Omega}\setminus\{0\}$. We are interested in closed extensions of this operator in $L^2(\overline{\Omega})$.

By introducing polar coordinates (x,θ) near 0, where $x\geq 0$ and $0\leq\theta\leq\alpha$, we blow up the origin and obtain a manifold $\overline{M}$ with corners of codimension two. The blow-down map takes $\overline{M}\to\overline{\Omega}$, $\partial_{\mathrm{sing}}\overline{M}\to 0$, and $\partial_{\mathrm{reg}}\overline{M}\setminus\partial_{\mathrm{sing}}\overline{M}\to\partial\overline{\Omega}\setminus\{0\}$.

Δ induces a cone operator on $\overline{M}$, and the boundary condition is the homogeneous Dirichlet boundary condition on $\partial_{\mathrm{reg}}\overline{M}$. The radial variable x gives rise to a defining function for $\partial_{\mathrm{sing}}\overline{M}$. Near $\partial_{\mathrm{sing}}\overline{M}$, we have $\Delta=x^{-2}\big((xD_x)^2+D^2_\theta\big)$. We will henceforth write Δ_{Dir} for this operator to emphasize that it is equipped with Dirichlet boundary conditions.

We consider Δ_{Dir} an unbounded operator in $x^{-1}L^2_b(\overline{M})$. Observe that the blow-down map takes this space to $L^2(\overline{\Omega})$, the space we are interested in.

The wealth of extensions of Δ_{Dir} depends strongly on the angle α. More precisely, if $0<\alpha<\pi$, then

$$\mathcal{D}_{\min}(\Delta_{\mathrm{Dir}})=\mathcal{D}_{\max}(\Delta_{\mathrm{Dir}})=\{u\in x^1H^2_b(\overline{M});\ u=0\text{ on }\partial_{\mathrm{reg}}\overline{M}\}.$$

If $\alpha=\pi$ (the case when the entire boundary of $\overline{\Omega}$ is smooth), then still

$$\mathcal{D}_{\min}(\Delta_{\mathrm{Dir}})=\mathcal{D}_{\max}(\Delta_{\mathrm{Dir}})\widehat{=}H^2(\overline{\Omega})\cap H^1_0(\overline{\Omega}),$$

but this space contains $\{u\in x^1H^2_b(\overline{M});\ u=0\text{ on }\partial_{\mathrm{reg}}\overline{M}\}$ as a proper subspace of infinite codimension. This provides another simple example that shows that the scale of weighted b-Sobolev spaces does not necessarily fit into the natural functional analytic framework of domains of closed extensions of an operator.

Consequently, whenever $0<\alpha\leq\pi$, we have $\mathcal{D}_{\min}(\Delta_{\mathrm{Dir}})=\mathcal{D}_{\max}(\Delta_{\mathrm{Dir}})$, the domain of the Friedrichs extension of the Laplacian. Thus the span of the eigenfunctions is dense by the spectral theorem.

The situation is more interesting for $\pi<\alpha<2\pi$. In this case,

$$\mathcal{D}_{\max}(\Delta_{\mathrm{Dir}})=\mathcal{D}_{\min}(\Delta_{\mathrm{Dir}})\oplus\mathrm{span}\{\omega(x)\varphi(\theta)x^{\pi/\alpha},\omega(x)\varphi(\theta)x^{-\pi/\alpha}\},$$

where $\omega \in C_c^\infty(\overline{\mathbb{R}}_+)$ is a cut-off function supported near the origin, and $\varphi(\theta) = \sin\big((\pi/\alpha)\theta\big)$ is an eigenfunction of D_θ^2 to the eigenvalue $(\pi/\alpha)^2$ on the interval $[0,\alpha]$ subject to Dirichlet boundary conditions (we are using polar coordinates here as above). Similarly to Example 5.2, we will show using Theorem 5.3 that the system of generalized eigenfunctions of Δ_{Dir} is complete in $x^{-1}L_b^2(\overline{M})$ for all domains $\mathcal{D} \subset \mathcal{D}_{\max}(\Delta_{\mathrm{Dir}})$ such that $\dim \mathcal{D}/\mathcal{D}_{\min}(\Delta_{\mathrm{Dir}}) = 1$. This includes infinitely many selfadjoint and, most importantly, nonselfadjoint extensions where the statement is nontrivial.

Clearly, Δ_{Dir} is c-elliptic with parameter $\lambda \in \Gamma$ for all rays $\Gamma \neq \overline{\mathbb{R}}_+$, and, likewise, the c-principal boundary symbol with parameter $\lambda \in \Gamma$ is invertible for all these rays Γ. In other words, the first two bulleted assumptions of Theorem 5.3 are satisfied for Δ_{Dir} for all rays $\Gamma \neq \overline{\mathbb{R}}_+$. In order to apply Theorem 5.3, we need to check the remaining assumptions on the normal operator. The normal operator is the positive Dirichlet Laplacian $\Delta_{\wedge,\mathrm{Dir}}$ on the angular domain V, written in polar coordinates. The L^2-realizations satisfy

$$\mathcal{D}_{\wedge,\max}(\Delta_{\wedge,\mathrm{Dir}}) = \mathcal{D}_{\wedge,\min}(\Delta_{\wedge,\mathrm{Dir}}) \oplus \operatorname{span}\{\omega(x)\varphi(\theta)x^{\pi/\alpha}, \omega(x)\varphi(\theta)x^{-\pi/\alpha}\}$$

as above. This induces an isomorphism

$$\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min} \cong \operatorname{span}\{\varphi(\theta)x^{\pi/\alpha}, \varphi(\theta)x^{-\pi/\alpha}\},$$

and the scaling action κ_ϱ on $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ takes the form

$$\kappa_\varrho\big(\varphi(\theta)x^{\pi/\alpha}\big) = \varrho^{\pi/\alpha} \cdot \varphi(\theta)x^{\pi/\alpha} \text{ and } \kappa_\varrho\big(\varphi(\theta)x^{-\pi/\alpha}\big) = \varrho^{-\pi/\alpha} \cdot \varphi(\theta)x^{-\pi/\alpha}$$

on the basis elements in the image of this isomorphism. Choose an arbitrary domain $\mathcal{D}_\wedge \subset \mathcal{D}_{\wedge,\max}$ with $\dim \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} = 1$. $\mathcal{D}_\wedge$ is represented by $\operatorname{span}\{a \cdot \varphi(\theta)x^{\pi/\alpha} + b \cdot \varphi(\theta)x^{-\pi/\alpha}\}$ for some $(a,b) \neq (0,0)$. In the Grassmannian of 1-dimensional subspaces of $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ we get

$$\begin{aligned} \kappa_\varrho\big(\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}\big) &= \operatorname{span}\{\varrho^{\pi/\alpha}a \cdot \varphi(\theta)x^{\pi/\alpha} + \varrho^{-\pi/\alpha}b \cdot \varphi(\theta)x^{-\pi/\alpha}\} \\ &= \operatorname{span}\{\varrho^{2\pi/\alpha}a \cdot \varphi(\theta)x^{\pi/\alpha} + b \cdot \varphi(\theta)x^{-\pi/\alpha}\} \underset{\varrho \to 0}{\longrightarrow} \begin{cases} \operatorname{span}\{\varphi(\theta)x^{\pi/\alpha}\} & \text{if } b = 0, \\ \operatorname{span}\{\varphi(\theta)x^{-\pi/\alpha}\} & \text{if } b \neq 0. \end{cases} \end{aligned}$$

It is easy to see that both domains $\mathcal{D}_{\wedge,\pm\alpha} = \mathcal{D}_{\wedge,\min}(\Delta_{\wedge,\mathrm{Dir}}) \oplus \operatorname{span}\{\omega(x)\varphi(\theta)x^{\pm\pi/\alpha}\}$ are selfadjoint for $\Delta_{\wedge,\mathrm{Dir}}$. Consequently, every ray $\Gamma \subset \mathbb{C}$ not parallel to the real line is a ray of minimal growth for $\Delta_{\wedge,\mathrm{Dir}}$ for all domains $\mathcal{D}_\wedge \subset \mathcal{D}_{\wedge,\max}$ with $\dim \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} = 1$. The reasoning here is completely analogous to Example 5.2.

Theorem 5.3 now applies, and we conclude that the system of generalized eigenfunctions of Δ_{Dir} is complete in $x^{-1}L_b^2(\overline{M})$ for all domains $\mathcal{D} \subset \mathcal{D}_{\max}(\Delta_{\mathrm{Dir}})$ with $\dim \mathcal{D}/\mathcal{D}_{\min}(\Delta_{\mathrm{Dir}}) = 1$ as was claimed.

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